On the Calculation of Magnetic Fields Based on Multipole Modeling of Focal Biological Current Sources

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ABSTRACT Spatially restricted biological current distributions, like the primary neuronal response in the human somatosensory cortex evoked by electric nerve stimulation, can be described adequately by a current multipole expansion. Here analytic formulas are derived for computing magnetic fields induced by current multipoles in terms of an nth-order derivative of the dipole field. The required differential operators are given in closed form for arbitrary order. The concept is realized in different forms for an expansion of the scalar as well as the dyadic Green's function, the latter allowing for separation of those multipolar source components that are electrically silent but magnetically detectable. The resulting formulas are generally applicable for current sources embedded in arbitrarily shaped volume conductors. By using neurophysiologically relevant source parameters, examples are provided for a spherical volume conductor with an analytically given dipole field. An analysis of the signal-to-noise ratio for multipole coefficients up to the octapolar term indicates that the lateral extent of cortical current sources can be detected by magnetoencephalographic recordings.

INTRODUCTION

Magnetoencephalography (MEG) uses superconducting quantum interference devices (SQUIDs) to measure the minute magnetic fields generated by currents in cortical neurons with a high signal-to-noise ratio. The "primary" (or "impressed") intraneuronal currents are accompanied by extracellular ("volume" or "return") currents that extend throughout the volume conductor (i.e., the head) and complete the current circuit. Commonly, the magnetic fields measured outside the volume conductor are explained by modeling the primary neuronal current distribution by an equivalent current dipole that indicates in particular the "center of mass" of the activated neuron population. The information content of high-precision MEG measurements, however, may allow for an even more detailed description of spatial features of the underlying current source distribution.

Depending on the expected source configuration, different source models are adequate, e.g., a sum of few dipoles, a dipole field (Mosher et al., 1992; Lütkenhöner et al., 1995), or a multipole expansion. While a sum of a few dipoles is appropriate for a small number of spatially well-separated sources consisting of essentially pointlike regions of activity, both the dipole field approach and the multipole expansion intend to describe a distributed source current density with an essential finite extent (Fig. 1). The crucial difference between these two approaches is found in the number of degrees of freedom: in a dipole field reconstruction, the measured data do not contain sufficient information to uniquely determine all of the dipole moments to be estimated. Hence the introduction of regularization terms (like minimum norm) is inherently necessary; accordingly, the reconstructed current density depends critically on the choice of the regularization parameter. In contrast, multipole coefficients can be uniquely determined if they are used to model those source components that produce a nonvanishing field outside the volume conductor (Sarvas, 1987). Thus, in view of reconstruction algorithms, the multipole approximation is mathematically close to the approximation of a few dipoles, but it describes extended sources, as does the dipole field approach.

Up to now, magnetic fields induced by current multipoles were calculated only for those cases for which the contributions from volume currents vanish, e.g., for the radial component of the magnetic field in the case of a spherical volume conductor (Nenonen et al., 1985) or for the normal component in the case of an infinite half-space (Erné et al., 1988; Haberkorn, 1994). Volume contributions of current quadrupoles for a spherical volume conductor were included only in Fieseler (1995).

In this paper we present a method for calculating the magnetic field, induced by a current multipole of arbitrary order embedded in arbitrarily shaped volume conductors, from a given dipole field. The method is rooted in the well-known observation (in the case of a scalar multipole expansion) that, for example, the magnetic field of a current quadrupole can be expressed as the difference between the fields of two identical current dipoles with opposite sign and with origins slightly shifted against each other. Because it is understood to take the limit of zero distance and infinite dipole strength, the resulting magnetic field is given by the derivative of a dipole field with respect to the dipole's origin.

This relation seems to be almost unexploited in the literature. Exceptions are the calculation of the electric potential of a current dipole as the derivative of the potential due to a charge monopole (de Munck, 1988; Zhang and Jewett,

Received for publication 20 September 1996 and in final form 19 May 1997.

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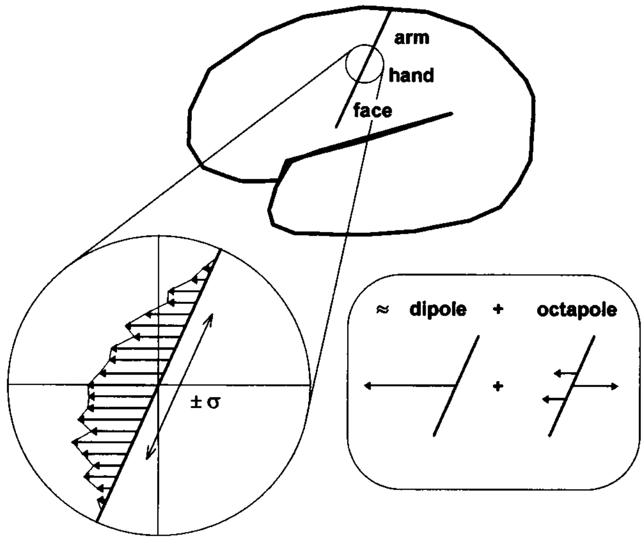

FIGURE 1 Multipolar description of brain current sources. The case of a focal neuronal activity can be studied in the hand respresentation area of the primary somatosensory cortex located in the anterior wall of the postcentral gyrus, where a circumscript neuronal response is evoked in Brodman area 3b pyramidal cells at ~20 ms after electric stimulation of the contralateral median nerve at the wrist. Because of the neuroanatomy of the activated cell population (with elongated parallel apical dendrites, which are oriented tangentially to the skull and orthogonally to the central sulcus), this focal current source may be reasonably approximated as a laterally extended, one-dimensional current distribution that can be modeled by a dipolar plus octapolar term of the multipole expansion (the quadrupolar term vanishes in the case of source symmetry). The octapole coefficient is related to the standard deviation σ of the source distribution.

1993) and the calculation of the magnetic field induced by the volume current of a current dipole as a derivative of a "monopole field" (Ferguson and Durand, 1991, 1992; Durand and Lin, 1997). Here we develop a theory valid for multipoles of arbitrary order as well as for arbitary volume conductors.

The multipole expansion of the scalar Green's function is conceptually simpler, and we present in the second section the general formalism for this representation. In the third section a modified version, consisting of an expansion of the dyadic Green's function, is deduced; this expansion is intrinsically free of redundancies and allows for a separation of electrically silent components. In the fourth section we elaborate on the physical interpretation of multipolar source descriptions, and in the fifth section we show fields for two instructive examples using neurophysiologically relevant current source parameters. We finally discuss the results from the perspective of biophysical applications in the sixth section.

SCALAR EXPANSION

General scalar multipole expansion

Using the common splitting of a total current $\vec{J}$ into a volume and a primary part, $\vec{J} = \vec{J}^V + \vec{J}^I$, the magnetic field $\vec{B}_{inf}$ induced by $\vec{J}^I$ in an infinite homogeneous volume conductor can be written as

$$\vec{B}_{inf}(\vec{r}) = \frac{\mu_0}{4\pi} \nabla \times \int dV' \frac{\vec{J}^I(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (1)$$

and a multipole expansion is defined through the expansion of the Green's function:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n,m} f_{n,m}(\vec{r}')\tilde{f}_{nm}(\vec{r}) \quad (2)$$

where $\tilde{f}_{nm}(\vec{r})$ is a function of order r^{-n-1}, and $f_{nm}(\vec{r}')$ is a function of order r'^n. $\vec{B}_{inf}(\vec{r})$ can now be written as

$$\vec{B}_{inf}(\vec{r}) = -\frac{\mu_0}{4\pi} \sum_{nm} \vec{a}_{nm} \times \nabla\tilde{f}_{nm}(\vec{r}) \quad (3)$$

where

$$\vec{a}_{nm} = \int dV \vec{J}^I(\vec{r}) f_{nm}(\vec{r}) \quad (4)$$

are the multipole coefficients of $\vec{J}^I$.

The magnetic field outside a volume conductor is computed by applying a solution operator $\vec{L}$ to $\vec{J}^I$. For example, if the volume conductor is a homogeneous sphere, the volume currents do not generate a radial magnetic field component, and thus $\vec{L}$ consists of the following procedure: compute the radial component of $\vec{B}_{inf}$, integrate from a given point $\vec{r}$ along the radial direction to infinity (which gives the magnetic scalar potential), and take the gradient to get the magnetic field $\vec{B}(\vec{r})$ (Sarvas, 1987). In short: $\vec{B} = \vec{L}(\vec{J}^I)$.

It is important to note that 1) $\vec{L}$ is linear and 2) $\vec{L}$ does not depend on $\vec{J}^I$ (it is *applied* to $\vec{J}^I$), particularly not on a multipole's position. These properties are generally valid and are not restricted to the spherical volume conductor.

Here we apply $\vec{L}$ to a "pure" multipole current, i.e., a current with a single nonvanishing multipole coefficient placed at $\vec{r}_0$, which is defined by the property

$$\int dV \vec{J}^I(\vec{r}) f_{nm}(\vec{r} - \vec{r}_0) = \vec{a}\delta_{mm_0}\delta_{nn_0} \quad \forall m, n \quad (5)$$

for some fixed m_0 and n_0.

Example: magnetic field of a quadrupolar current source

As a first step beyond the dipole we take the term containing f_{1-1} from the quadrupole (Katila, 1983; Fieseler, 1995), which is defined through $f_{1-1}(\vec{r}) = y$, $f_{10}(\vec{r}) = z$, and $f_{11}(\vec{r}) = x$, and we let $\vec{a}_{1-1}$ point in the x direction with magnitude a. Because the dipole is defined through $f_{00}(\vec{r}) = 1$ and all other multipoles contain higher order polynomials, partial integration in Eq. 5 shows that the quadrupole current can be given explicitly by

$$\vec{J}^{\mathrm{I}}(\vec{r}) = -\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \frac{\partial}{\partial y} \delta^3(\vec{r} - \vec{r}_0) = +\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \frac{\partial}{\partial y_0} \delta^3(\vec{r} - \vec{r}_0) \tag{6}$$

The magnetic field can now be calculated as

$$\begin{aligned} \vec{B}_{\mathrm{quad}} &= \vec{L}(\vec{J}^{\mathrm{I}}) = \vec{L}\left(\frac{\partial}{\partial y_0} \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \delta^3(\vec{r} - \vec{r}_0) \right) \\ &= \frac{\partial}{\partial y_0} \vec{L}\left(\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \delta^3(\vec{r} - \vec{r}_0) \right) = \frac{\partial}{\partial y_0} \vec{B}_{\mathrm{dip}} \end{aligned} \tag{7}$$

where $\vec{L}$ is now applied to a pure current dipole in the x direction at position $\vec{r}_0$ with a solution that is well known in the case of a spherical volume conductor (Ilmoniemi et al., 1985; Sarvas, 1987). (We emphasize that the notions "current dipole" and its "magnetic field" $\vec{B}_{\mathrm{dip}}$ in this context are only formal, because these expressions do not have the dimensions of a current dipole or a magnetic field, respectively. The reason is that the "dipole moment" a is physically a quadrupole moment. However, mathematically the argument of $\vec{L}$ is proportional to a current dipole with a dimensionful proportionality constant. Because of the linearity of $\vec{L}$, the resulting "magnetic" field will also be proportional to a "real" magnetic field with the same proportionality constant. Differentiation with respect to y_0 at the end of the calculation will result in a field with proper dimensions; this kind of formal notion occurs throughout the paper.)

General case: magnetic field of multipolar current sources

To express the magnetic fields in general as (more complex) derivatives of a dipole field, one must find the "dual" differential operators $\hat{f}_{\mathrm{nm}}(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, which fulfill the "orthonormality relation"

$$\hat{f}_{\mathrm{nm}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_{\tilde{n}\tilde{m}}(x, y, z)|_{x=y=z=0} = \delta_{\mathrm{m}\tilde{\mathrm{m}}}\delta_{n\tilde{n}} \tag{8}$$

This notion of "orthogonality" is equivalent to the usage by Wikswo and Swinney (1984), in which multipole charges up to octapole order were constructed as a limit of differences of charge monopoles.

Because f_{nm} is a polynomial of degree n with the general form

$$f_{\mathrm{nm}}(x, y, z) = \sum_{\mathrm{kl}} a_{\mathrm{kl}}^{\mathrm{nm}}\, x^{\mathrm{k}} y^{\mathrm{l}} z^{\mathrm{n-k-l}} \tag{9}$$

$\hat{f}_{\mathrm{nm}}$ must be a sum of nth-order partial derivatives. As a trial ansatz for the operators, we choose

$$\hat{f}_{\mathrm{nm}}^{\mathrm{trial}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = f_{\mathrm{nm}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{10}$$

i.e., the Cartesian coordinates are replaced by the respective derivatives. Inserting the general form (Eq. 9) into Eq. 8 gives

$$\begin{aligned} &\hat{f}_{\mathrm{nm}}^{\mathrm{trial}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_{\mathrm{nm}}(x, y, z)|_{x=y=x=z=0} \\ &\qquad = \sum_{\mathrm{kl}} k!\,l!\,(n-k-l)!\,(a_{\mathrm{kl}}^{\mathrm{nm}})^2 > 0 \end{aligned} \tag{11}$$

and

$$\hat{f}_{\mathrm{nm}}^{\mathrm{trial}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_{\tilde{n}\tilde{m}}(x, y, z)|_{x=y=z=0} = 0 \quad \forall n \neq \tilde{n} \tag{12}$$

Equation 11 also holds if one takes linear combinations of f_{nm} (and $\hat{f}_{\mathrm{nm}}$) into account with new coefficients $\tilde{a}_{\mathrm{kl}}^{\mathrm{nm}}$, showing the independence of the set of operators. Thus the trial operators form a basis in the operator space. According to Eq. 12, the final $\hat{f}_{\mathrm{nm}}$ can in general be expressed as linear combinations of those $\hat{f}_{\mathrm{nm}}^{\mathrm{trial}}$ that are of the same order, and in the worst case for a given order, one is left with a finite algebraic problem.

Once these dual operators are found, partial integration in Eq. 5 shows that a pure multipole current at position $\vec{r}_0$ can be written as

$$\begin{aligned} \vec{J}^{\mathrm{I}}(\vec{r}, \vec{r}_0) &= (-1)^{n_0}\, \vec{a}\hat{f}_{\mathrm{n_0m_0}}(\nabla)\delta^3(\vec{r} - \vec{r}_0) \\ &= \vec{a}\hat{f}_{\mathrm{n_0m_0}}(\nabla_0)\delta^3(\vec{r} - \vec{r}_0) \end{aligned} \tag{13}$$

and analogously to the quadrupolar example above, the corresponding magnetic field is

$$\vec{B}(\vec{r}) = \hat{f}_{\mathrm{n_0m_0}}(\nabla_0)\vec{B}_{\mathrm{dip}}(\vec{a}, \vec{r}_0, \vec{r}) \tag{14}$$

The operators ∇ and ∇_0 differentiate with respect to the coordinates $\vec{r}$ and $\vec{r}_0$, respectively.

Expansion in spherical harmonics

Here we study the expansion of the scalar Green's function in spherical harmonics ($Y_{\mathrm{n}}^{\mathrm{m}}$) (Geselowitz, 1965; Burghoff et al., 1991)

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\mathrm{n=0}}^{\infty} \sum_{\mathrm{m=0}}^{\mathrm{n}} \frac{f_{\mathrm{nm}}^{\mathrm{R}}(\vec{r})}{|\vec{r}|^{2n+1}} f_{\mathrm{nm}}^{\mathrm{R}}(\vec{r}') + \frac{f_{\mathrm{nm}}^{\mathrm{I}}(\vec{r})}{|\vec{r}|^{2n+1}} f_{\mathrm{nm}}^{\mathrm{I}}(\vec{r}') \tag{15}$$

with $f_{\mathrm{nm}}^{\mathrm{R}} + if_{\mathrm{nm}}^{\mathrm{I}} = r_{\mathrm{n}} Y_{\mathrm{n}}^{\mathrm{m}}$. Getting the form of Eq. 2 is a matter of notation.

It is shown in Appendix A that this representation is already orthogonal in the sense of Eq. 8 (with $\hat{f} = \hat{f}^{\mathrm{trial}}$), and the properly normalized dual operators can be given in closed form ($(2n-1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$) as

$$\hat{f}^{\mathrm{R,I}}_{\mathrm{nm}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)=\frac{(2-\delta_{\mathrm{m0}})\cdot(n-m)!}{(n+m)!(2n-1)!!}f^{\mathrm{R,I}}_{\mathrm{nm}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) \tag{16}$$

or, equivalently, as

$$\hat{f}^{\mathrm{R}}_{\mathrm{nm}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)+i\hat{f}^{\mathrm{I}}_{\mathrm{nm}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)=\frac{(2-\delta_{\mathrm{m0}})}{(n+m)!}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^{\mathrm{m}}\left(\frac{\partial}{\partial z}\right)^{\mathrm{n-m}} \tag{17}$$

Summary

We can express the magnetic field $\vec{B}(\vec{r})$ due to an arbitrary current $\vec{J}^{\mathrm{I}}$ with multipole coefficients $\vec{a}_{\mathrm{nm}}+i\vec{b}_{\mathrm{nm}}$ at position $\vec{r}_0$,

$$\vec{a}_{\mathrm{nm}}+i\vec{b}_{\mathrm{nm}}=\int dV\,\vec{J}^{\mathrm{I}}(\vec{r})(f^{\mathrm{R}}_{\mathrm{nm}}(\vec{r}-\vec{r}_0)+if^{\mathrm{I}}_{\mathrm{nm}}(\vec{r}-\vec{r}_0)) \tag{18}$$

as the sum of all multipole fields,

$$\vec{B}(\vec{r})=\mathrm{Re}\left(\sum_{\mathrm{n=0}}^{\infty}\sum_{\mathrm{m=0}}^{\mathrm{n}}\frac{(2-\delta_{\mathrm{m0}})}{(n+m)!}\left(\frac{\partial}{\partial x_0}-i\frac{\partial}{\partial y_0}\right)^{\mathrm{m}}\left(\frac{\partial}{\partial z_0}\right)^{\mathrm{n-m}}(\vec{B}_{\mathrm{dip}}(\vec{a}_{\mathrm{nm}},\vec{r},\vec{r}_0)+i\vec{B}_{\mathrm{dip}}(\vec{b}_{\mathrm{nm}},\vec{r},\vec{r}_0))\right) \tag{19}$$

where $\vec{B}_{\mathrm{dip}}(\vec{c}_{\mathrm{nm}},\vec{r},\vec{r}_0)$ is the magnetic field at point $\vec{r}$ due to a current dipole of magnitude $\vec{c}_{\mathrm{nm}}$ at position $\vec{r}_0$. Note that the differential operator is complex conjugated to give positive contributions from the imaginary parts. (Again, $\vec{B}_{\mathrm{dip}}$ does not have the dimension of a magnetic field; only the final result has.)

This scalar representation, however, is flawed by redundancies. For example, a current with coefficients $\vec{a}_{10}=J\hat{e}_z$, $\vec{a}_{11}=J\hat{e}_x$, and $\vec{b}_{11}=J\hat{e}_y$ produces no field. This is usually cured by taking only a reduced number of linear combinations of multipole coefficients into account (Wikswo and Swinney, 1984), which is a basis transformation and can in general spoil the orthogonality relation (Eq. 8). If one describes the basis transformation $f_{\mathrm{nm}}\rightarrow g_{\mathrm{nm}}$ with $(2n+1)\times(2n+1)$ block matrices A^{n} ($g_{\mathrm{nm}}=\Sigma_{\mathrm{l}}A^{\mathrm{n}}_{\mathrm{ml}}f_{\mathrm{nl}}$), then the operators transform with $B^{\mathrm{n}}:=(A^{\mathrm{n}^{-1}})^{\mathrm{T}}$ ($\hat{g}_{\mathrm{nm}}=\Sigma_{\mathrm{l}}B^{\mathrm{n}}_{\mathrm{ml}}\hat{f}_{\mathrm{nl}}$). Now those coefficients with zero magnetic field can be omitted.

DYADIC EXPANSION

Formulation of the expansion

An alternative multipole representation, which makes it possible to separate electrically silent source components, is given by the expansion of the dyadic Green's function (Grynszpan and Geselowitz, 1973; Haberkorn et al., 1992)

$$\vec{B}_{\mathrm{inf}}=\frac{\mu_0}{4\pi}\mathrm{Re}\left(\sum_{\mathrm{n=1}}^{\infty}\sum_{\mathrm{m=0}}^{\mathrm{n}}(A_{\mathrm{nm}}+iB_{\mathrm{nm}})\vec{r}\times\nabla\frac{-\bar{Y}^{\mathrm{m}}_{\mathrm{n}}}{nr^{\mathrm{n+1}}}\right)+\frac{\mu_0}{4\pi}\mathrm{Re}\left(\sum_{\mathrm{n=1}}^{\infty}\sum_{\mathrm{m=0}}^{\mathrm{n}}(\alpha_{\mathrm{nm}}+i\beta_{\mathrm{nm}})\nabla\frac{-\bar{Y}^{\mathrm{m}}_{\mathrm{n}}}{r^{\mathrm{n+1}}}\right)\equiv\vec{B}^{\mathrm{E}}_{\mathrm{inf}}+\vec{B}^{\mathrm{M}}_{\mathrm{inf}} \tag{20}$$

with "electric" multipole coefficients

$$A_{\mathrm{nm}}+iB_{\mathrm{nm}}=g_{\mathrm{nm}}\int dV\,\vec{J}\cdot\nabla Y^{\mathrm{m}}_{\mathrm{n}}r^{\mathrm{n}} \tag{21}$$

and "magnetic" multipole coefficients

$$\alpha_{\mathrm{nm}}+i\beta_{\mathrm{nm}}=\frac{g_{\mathrm{nm}}}{n+1}\int dV\,(\vec{r}\times\vec{J})\cdot\nabla Y^{\mathrm{m}}_{\mathrm{n}}r^{\mathrm{n}} \tag{22}$$

where $g_{\mathrm{nm}}=(2-\delta_{\mathrm{m0}})[(n-m)!/(n+m)!]$.

$\vec{B}_{\mathrm{inf}}$ is again the magnetic field for an infinite homogeneous conductor. The "magnetic" components are already written as a gradient of a scalar function. They are always curl free, are not affected by the volume conductor (hence $\vec{B}^{\mathrm{M}}=\vec{B}^{\mathrm{M}}_{\mathrm{inf}}$), and cannot be observed electrically. The notion "magnetic" refers to this property, and we will only have to analyze the "electric" components.

Notably, the magnetic fields generated by the "electric" multipole components cannot be written as derivatives of dipole fields as in the second section, above. For example, a difference between two current dipoles will in general contain a "magnetic" source component wherever we put the expansion point. Partial integration in Eq. 21 shows that the expansion is not based on $\vec{J}^{\mathrm{I}}$ but on $\nabla\vec{J}^{\mathrm{I}}$, but neither $\vec{J}^{\mathrm{I}}$ nor the magnetic field is uniquely determined by the knowledge of $\nabla\vec{J}^{\mathrm{I}}$. We thus have to formulate the theory solely in terms of $\nabla\vec{J}^{\mathrm{I}}$; this can be done by splitting the total magnetic field into a primary current part and a volume current part, where the latter depends only on $\nabla\vec{J}^{\mathrm{I}}$ but not on $\nabla\times\vec{J}^{\mathrm{I}}$. The magnetic field induced by a volume current can then generally be expressed analogously to the previous section as $\vec{L}(\nabla\vec{J}^{\mathrm{I}})$, where $\vec{L}$ is the linear solution operator for this problem.

Monopole field

In principle one could now compute the volume contribution for a current dipole, and the volume contribution of a current multipole ("multipole volume field") could then be expressed as a derivative of the former. However, this can be simplified significantly because for the dipole volume fields, according to the nonzero coefficients $A_{1\mathrm{m}}+iB_{1\mathrm{m}}=\int dV(-\nabla\vec{J}^{\mathrm{I}})rY^{\mathrm{m}}_{1}$, where $rY^{0}_{1}=z$ and $rY^{1}_{1}=x+iy$, the source $-\nabla\vec{J}$ (now evaluated at position $\vec{r}_0$) is itself given as a derivative:

$$-\nabla\vec{J}=-\left(A_{10}\frac{\partial}{\partial z}+A_{11}\frac{\partial}{\partial x}+B_{11}\frac{\partial}{\partial y}\right)\delta^3(\vec{r}-\vec{r}_0) \tag{23}$$

By applying the same reasoning as in the second section above, the magnetic dipole field can now be written as a derivative of a "monopole field" $\vec{B}_{\mathrm{mon}}$, e.g.,

$$\vec{L}(-\nabla\vec{J}^{\mathrm{I}}) = \vec{L}\left(A_{11}\frac{\partial}{\partial x_0}\delta^3(\vec{r}-\vec{r}_0)\right)$$
$$= A_{11}\frac{\partial}{\partial x_0}\vec{L}(\delta^3(\vec{r}-\vec{r}_0)) = \frac{\mu_0 A_{11}}{4\pi}\frac{\partial}{\partial x_0}\vec{B}_{\mathrm{mon}} \quad (24)$$

However, $\vec{L}$ cannot be applied to a charge monopole, because a net charge different from zero implies a net flux of the electric field out of the volume conductor different from zero, which is inconsistent with the boundary conditions. Thus here we will define $\vec{L}(\delta^3(\vec{r}-\vec{r}_0))$ (and hence $\vec{B}_{\mathrm{mon}}$) by the property that its derivatives give the respective dipole volume fields. The existence follows from the fact that each component of $\vec{L}(-\nabla\vec{J}^{\mathrm{I}})$ is "curl free," e.g.,

$$\frac{\partial}{\partial x_0}\vec{L}\left(\frac{\partial}{\partial y_0}\delta^3(\vec{r}-\vec{r}_0)\right) = \frac{\partial}{\partial y_0}\vec{L}\left(\frac{\partial}{\partial x_0}\delta^3(\vec{r}-\vec{r}_0)\right) \quad (25)$$

We found the monopole field defined above to be identical to the one introduced by Ferguson and Durand (Ferguson and Durand, 1991, 1992; Durand and Lin, 1997); there are, however, formal differences. First, the monopole field appeared here naturally in the context of multipole fields used in the expansion of the dyadic Green's function. Second, its existence is shown formally, which is more compact than the physical approach of Ferguson and Durand (1992). Third, the final formula for the monopole field in a spherical volume conductor (Eq. B5) includes rotations to arbitrary monopole positions explicitly showing its symmetry.

The monopole field was defined through its derivatives with respect to $\vec{r}_0$, and hence it is unique only up to an additive function of $\vec{r}$. A convenient way to fix this function is to choose $\vec{B}_{\mathrm{mon}}(\vec{r},\vec{R}_0) = 0$ for an arbitrary reference point $\vec{R}_0$. $\vec{B}_{\mathrm{mon}}(\vec{r},\vec{r}_0)$ then corresponds to the magnetic field of the volume current induced by a monopole with positive charge at $\vec{r}_0$ and a monopole with negative charge at $\vec{R}_0$ (Ferguson and Durand, 1992).

In Appendix B we show that $\vec{B}_{\mathrm{mon}}$ for the spherical volume conductor with the reference point placed in the center of the sphere is given by

$$\vec{B}_{\mathrm{mon}}(\vec{r},\vec{r}_0) = \frac{\vec{r}_0\times\vec{r}}{r^2r_0^2-(\vec{r}\cdot\vec{r}_0)^2}\left(\frac{\vec{r}\cdot\vec{r}_0-\vec{r}_0^{\,2}}{|\vec{r}-\vec{r}_0|}-\frac{\vec{r}\cdot\vec{r}_0}{r}\right) \quad (26)$$

Summary

For multipole coefficients $A_{\mathrm{nm}} + iB_{\mathrm{nm}}$, the source $-\vec{\nabla}\vec{J}^{\mathrm{I}}$ is given by

$$-\nabla\vec{J}^{\mathrm{I}} = \left(\frac{A_{\mathrm{nm}}}{g_{\mathrm{nm}}}\hat{f}^{\mathrm{R}}_{\mathrm{nm}}(\nabla_0) + \frac{B_{\mathrm{nm}}}{g_{\mathrm{nm}}}\hat{f}^{\mathrm{I}}_{\mathrm{nm}}(\nabla_0)\right)\delta^3(\vec{r}-\vec{r}_0) \quad (27)$$

The operators $\hat{f}^{\mathrm{R,I}}_{\mathrm{nm}}(\vec{\nabla}_0)$ can be drawn in front of $\vec{L}$, which itself is (formally) applied to a monopole source $\delta^3(\vec{r}-\vec{r}_0)$. The result is the volume current contribution to which the primary current contribution has to be added, finally ending up with

$$\vec{B} = \frac{\mu_0}{4\pi}\mathrm{Re}\left(\sum_{n=1}^{\infty}\sum_{m=0}^{n}(A_{\mathrm{nm}}+iB_{\mathrm{nm}})\left(\vec{B}^{\mathrm{E}}_{\mathrm{nm}} + \frac{1}{(n-m)!}\left(\frac{\partial}{\partial x_0}-i\frac{\partial}{\partial y_0}\right)^{m}\left(\frac{\partial}{\partial z_0}\right)^{n-m}\vec{B}_{\mathrm{mon}}(\vec{r},\vec{r}_0)\right)\right) + \vec{B}^{\mathrm{M}} \quad (28)$$

where

$$\vec{B}^{\mathrm{E}}_{\mathrm{nm}} = \vec{r}\times\nabla\frac{-\bar{Y}^{\mathrm{m}}_{\mathrm{n}}}{nr^{n+1}}.$$

PHYSICAL INTERPRETATION OF MULTIPOLAR SOURCE DESCRIPTIONS

It is the central feature of the multipolar source description to reconstruct only those aspects that are backed by data: by setting an upper bound for the order of multipole expansion, it does not try to reconstruct a priori unresolvable details of the current distribution.

For illustration, we will consider a primary current with fixed direction distributed on a straight line with coordinate ξ. The primary current can be written most generally as $\vec{J}(\xi) = \vec{a}g(\xi)$ with fixed vector $\vec{a}$ and a scalar function g. Because the current distribution, and accordingly also the expansion, is one-dimensional, there is only one multipole for each order. After omission of the index m, the polynomials f_{n} (see Eq. 2) and the respective dual operators $\hat{f}_{\mathrm{n}}$ are given by

$$f_{\mathrm{n}}(\xi) = \xi^{\mathrm{n}} \quad (29)$$

and

$$\hat{f}_{\mathrm{n}}\left(\frac{\partial}{\partial\xi}\right) = \frac{1}{n!}\frac{\partial^{\mathrm{n}}}{\partial\xi^{\mathrm{n}}}. \quad (30)$$

Choosing the coordinate ξ such that the expansion point is at $\xi = 0$, the multipole coefficients are

$$a_{\mathrm{n}} = \int d\xi g(\xi)\xi^{\mathrm{n}} \quad (31)$$

The full magnetic field can be calculated with Eq. 14:

$$\vec{B}(\vec{r}) = \sum_{n=0}^{\infty}\frac{a_{\mathrm{n}}}{n!}\frac{\partial^{\mathrm{n}}}{\partial\xi^{\mathrm{n}}}\vec{B}_{\mathrm{dip}}(\vec{a},\vec{r},\xi)|_{\xi=0} \quad (32)$$

where $\vec{B}_{\mathrm{dip}}$ is the dipole magnetic field due to the dipole moment $\vec{a}$.

This example may be regarded as a special case of the three-dimensional Taylor expansion of the Green's function

(Wikswo and Swinney, 1984), with a coordinate system that includes the straight line as one axis. All terms that contain the two coordinates orthogonal to ξ may be ignored because the respective multipole coefficients vanish identically.

Assuming in a first approximation that $g(\xi)$ does not change sign and that it is symmetrical around $\xi = 0$, the multipole coefficients (Eq. 31) can be interpreted as statistical moments of the stochastic variable ξ with (nonnormalized) probability density $g(\xi)$. Accordingly, the dipole moment corresponds to the normalization, the quadrupole moment vanishes (because of the symmetry), and the octapole moment is related to the variance (cf. Fig. 1). The lateral spatial extent of the current density may reasonably defined by the standard deviance of ξ:

$$\sigma = \sqrt{\mathrm{var}(\xi)} = \sqrt{a_2/a_0} \tag{33}$$

Hence the multipole expansion can provide a meaningful expression for the spatial source extent, which does not explicitly refer to eventually unresolvable details of the current distribution.

If the symmetry requirement is relaxed, the expansion point must be put at the "center of mass" of $g(\xi)$ to keep Eq. 33 valid. This corresponds to the vanishing of the quadrupole coefficient. In principle, the center of mass can be found by a multipole fit with omission of the quadrupole term. Up to octapole order, the procedure will then be identical to the symmetrical case.

As long as the multipole expansion is cut at some fixed order, only an approximate reconstruction can be obtained. The reason is that higher order corrections are in general not orthogonal to the low-order fields. Their parallel components are erroneously ascribed to the respective reconstructed low-order coefficients. For example, the result of a dipole fit is not the true equivalent current dipole but an approximation, which is necessarily distorted by higher order source terms. Similarly, a fit up to octapolar order will not provide the true spatial extent, even for noiseless data. However, the approximation can be expected to be very good; e.g., given the brain source model of the next section, the estimated spatial extent would be only 1.5% too small.

Restricting the source model, like cutting a multipole series, is adequate when an inverse calculation does not have a unique solution. For example, this occurs for a finite number of sensors, but can also occur for an infinite number of sensors with finite size, as in Wikswo and Roth (1988), Roth et al. (1989), and Tan et al. (1990). To make the inverse calculation stable, the authors introduced spatial low-pass filtering, which can be related to the multipole expansion. To see this, the current density $g(\xi)$ can be rewritten as a sum of pure current multipoles (see Eq. 13):

$$g(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{n!} \frac{\partial^n}{\partial \xi^n} \delta(\xi) \tag{34}$$

The Fourier transform $\hat{g}(k) = \int d\xi e^{-ik\xi} g(\xi)$ is given by

$$\hat{g}(k) = \sum_{n=0}^{\infty} \frac{a_n}{n!} (-i)^n k^n \tag{35}$$

which is the Taylor series of $\hat{g}(k)$ around $k = 0$.

In correspondence to a low-pass filter, which keeps only low-frequency components, the lower terms of a Taylor expansion provide information on the low-frequency behavior. It should be pointed out that a Taylor expansion of a function $\hat{g}(k)$ to finite order is generally a bad approximation for large k: an integrable function $\hat{g}(k)$ will be approximated by a function that diverges for $k \to \infty$. Analogously, cutting the expansion in Eq. 34 at arbitrary but finite order always results in an approximation of $g(\xi)$ that is strictly local and cannot be square integrable, although the original $g(\xi)$ might be.

The truncated multipole expansion is not supposed to approximate the current density as a whole function. The multipole (or Taylor) coefficients only describe bulk properties of the current density as discussed above. It is useful to visualize pure multipoles, as was done quite generally for electric potentials in (Wikswo and Swinney, 1984) and for magnetic fields and electric potentials in (Wikswo and Swinney, 1985). An expansion of the Green's function in spherical harmonics was used in the latter paper, where an effective theory for the forward calculation of magnetic fields was used. Formally speaking, this multipole expansion is "purely magnetic" or "purely electric" in the sense that the fields only depend on sources and sinks of $\vec{r} \times \vec{J}^i$ or $\vec{J}^i$, respectively. This led to the possibility of alternative configurations (A and B) as illustrated in Fig. 2. The quadrupole currents of A and B have the same sources and sinks; thus they have the same electric quadrupole moments of the dyadic multipole expansion. In the dyadic expansion presented in the third section above (Dyadic Expansion), configurations A and B are not pure multipoles, because both configurations have, in addition, nonvanishing magnetic multipole moments with opposite signs. Only for the average of A and B (i.e., configuration C) do all magnetic multipole coefficients vanish.

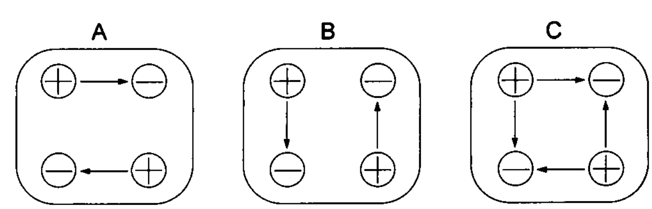

FIGURE 2 Current configurations representing a quadrupole in the expansion of the dyadic Green's function. If only electric coefficients are taken into account, A and B are alternative configurations with the same multipole coefficients. However, only averaging A and B, leading to configuration C, results in a pure electric current quadrupole, a configuration with vanishing magnetic multipole components.

EXAMPLES

All example fields presented here were calculated using a spherical volume conductor as a first-order approximation for the head for which the dipole (and monopole) field is analytically given. Calculation of the derivatives is straightforward. However, the resulting expressions, computed with the program Maple, are in general too lengthy to be presented explicitly. We checked the found formulas for the case of a spherical volume conductor. The magnetic field must fulfill two conditions: 1) it must be curl free, and 2) the radial component must be equal to the corresponding field component in the case of an infinite homogeneous volume conductor.

For the scalar expansion in the second section above (Scalar Expansion), the first condition is fulfilled, because the magnetic field was derived from a curl-free field by application of a differential operator that commutes with the curl. In fact, the whole derivation could have been done equally well with the magnetic scalar potentials. The second condition was checked explicitly for the quadrupolar example field above.

The situation is quite the opposite for the dyadic expansion. The second condition is fulfilled, because the volume contribution was separated, and, because of the cross-product with $\vec{r}$ in Eq. 26, it cannot generate a radial component. We checked the first condition for all multipoles up to the octapole.

For a first illustration we plotted the z component of magnetic fields induced by three current multipoles in a spherical volume conductor and compared them to the results in a half-space (Fig. 3). The effect of the spherical volume conductor is mainly a scale and magnitude transformation; this is expected because the sphere is an axially symmetrical deformation of the half-space. However, in case of the octapole field (A_{31}), where the two side extrema are larger, whereas all other extrema are smaller for the spherical volume conductor than for the half-space conductor, it becomes evident that the magnitude transformation does not follow a simple rule.

As a second example we discuss a "realistic" case in some detail. Similar computations were made by Okada (1985) and Pelizzone and Hari (1986), however, without describing corrections to a dipole field as higher order multipole fields. We numerically calculated the magnetic field of a "typical" neocortical source with finite lateral extent, which was constructed as a model for the current distribution in the hand area of the primary somatosensory cortex evoked 20 ms after electric median nerve stimulation (N20, Fig. 1; cf. Hari et al., 1993); it was approximated by 400 parallel dipoles (pointing in the x direction) with a total dipole moment $J = 10$ nAm, uniformly distributed along a line (elongated in the y direction) of length $d = 2$ cm with its center at the point $x = y = 0$ and $z = 6.5$ cm (the center of the spherical volume conductor is the origin of the coordinate system), and compared the result to the multipole expansion (Fig. 4). We assumed a sensor configuration measuring the z component of the magnetic field, again in the plane $z = 10.5$ cm.

The octapole coefficients for the dyadic product expansion read $A_{31} = -Jd^2/48$, $A_{33} = -Jd^2/96$, and $\beta_{21} = -Jd^2/36$, where $J = A_{11}$ is (the x component of) the dipole moment. However, for this case it is easier to use the expansion of the scalar Green's function as in the preceding section, with $a_n = 2^{-n}d^{n+1}/(n+1)$ for even n and $a_n = 0$ for odd n.

Fig. 4 shows that the convergence of the multipole series is very fast: the first nonvanishing (the octapolar) correction term to the dipolar approximation reduces the relative field

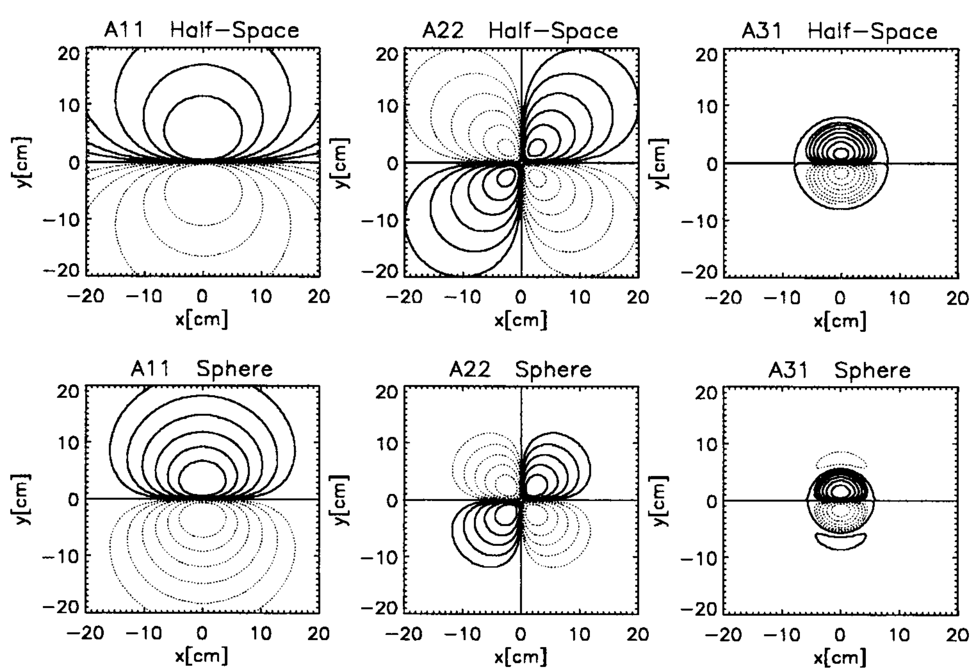

FIGURE 3 Contour maps of the z component of magnetic fields in the plane $z = 10.5$ cm generated by current multipoles embedded in a half-space (*upper row*) and a spherical volume conductor (*lower row*). The dipolar, quadrupolar, or octapolar current sources were placed at position $x = y = 0$ and $z = 6.5$ cm (center of sphere at $x = y = z = 0$). The multipole moments are arbitrary but are equal for sphere and half-space. Starting with the first nonzero contour level, which was chosen to be equal for the two volume conductors, values of adjacent isofield lines increase by a factor of 2.

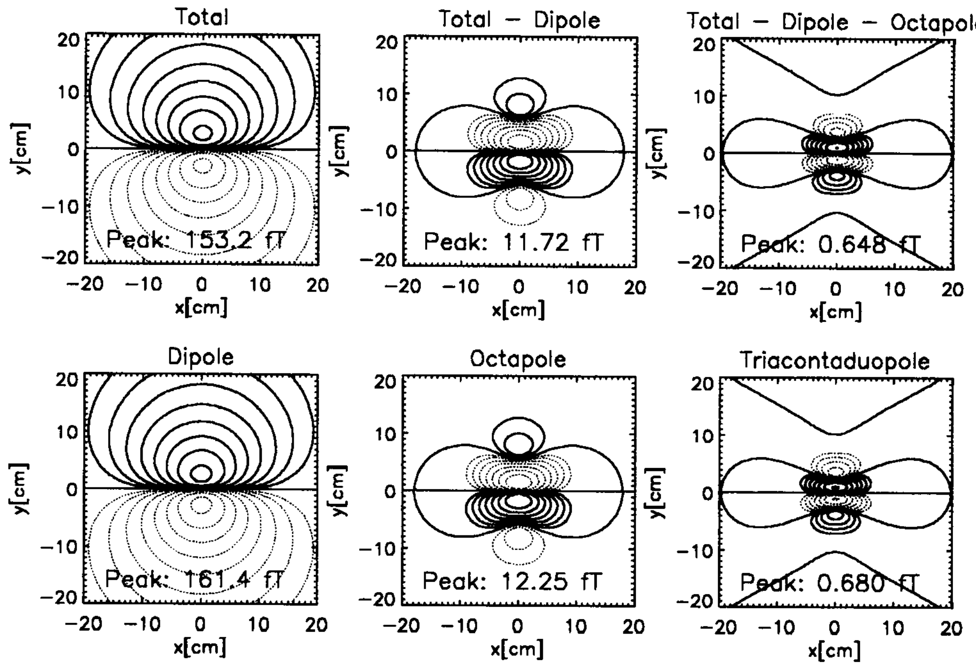

FIGURE 4 A scalar multipole expansion was used to calculate the z component of magnetic fields generated by multipole components of a current source with a 2 cm lateral extent located 4 cm below the measurement plane in a spherical volume conductor (cf. Fig. 3). The lower row of the figure shows contour maps of the individual contributions of the zeroth-, second-, and fourth-order multipoles to the z component of the magnetic field; they can be compared to the total (numerically calculated) field, with multipole fields subtracted as indicated in the titles (*upper row*). Adjacent nonzero contour levels differ by a factor of 2, where the first nonzero level has value 1 fT, 0.1 fT, and 0.01 fT for order 0, 2, and 4, respectively.

deviance (rms of differences between the numerically calculated total field and its approximation divided by rms of total field) by a factor of 25 from 4.4% (total, dipole) to 0.17% (total, dipole-octapole). Values for a source extent d that is not too large can be estimated from scaling arguments to $4.4\% \cdot d^2/4\ \mathrm{cm}^2$ and $0.17\% \cdot d^4/16\ \mathrm{cm}^4$. One reason for the high convergence rate is that because of the symmetry of the current distribution, the sum in Eq. 32 does not contain odd terms; for a comparable asymmetrical current distribution, one can expect an error reduction by factor of $\sim$5 for each additional multipole order.

Finally, we estimate whether an octapole coefficient can be measured for idealized geometric but realistic noise conditions. We assume that the true current distribution has the form described above, with the extent d being the only unknown parameter. By inspection of Fig. 4 one can recognize that the octapole field component might be difficult to detect because it shares structural features with the dipole field. In fact, only the part of the octapole field orthogonal to the dipole field can be used to discriminate it from the dipole, where "orthogonality" of two measured fields $B_1(n)$ and $B_2(n)$ ($n = 1, \ldots,$ number of channels) is defined by the scalar product $\langle B_1, B_2 \rangle = \Sigma_n B_1(n) B_2(n)$. Denoting the normalized, orthogonal part of the octapole field as $\hat{B}_{\mathrm{okt}}^{\perp}$, we estimate the "effective" octapole field from the projection of the measured data B onto $\hat{B}_{\mathrm{okt}}^{\perp}$. For a realistic sensor configuration consisting of 49 planar channels (as used, e.g., in Curio et al., 1995), the amplitude A of this projected field, $A = \langle B, \hat{B}_{\mathrm{okt}}^{\perp} \rangle$, has a value of 8 fT (for $d = 2$ cm) with a noise level equal to that of an individual channel; generally, A is proportional to d^2. Notably, a peak-to-peak noise level of 4 fT is already attainable in measurements of the somatosensory evoked magnetic fields for appropriately bandpassed data (Curio et al., 1994a,b).

DISCUSSION

We presented explicit formulas to calculate a magnetic field generated by an arbitrary multipole current embedded in an arbitrarily shaped volume conductor as the derivative of either a dipole or a monopole field. This allows for a detailed analysis of magnetic fields induced by multipole currents in the brain, which is tedious to compute by conventional integration techniques, which, moreover, are applicable only when assuming a highly symmetrical (e.g., spherical or half-space) volume conductor. In contrast, differentiation is computationally straightforward and can be easily implemented in a computer program for either analytically given or numerically approximated dipole fields in the case of arbitrarily shaped volume conductors. If the expansion of the dyadic Green's function is used for the numerical case, it is not necessary to compute the monopole field explicitly, because all multipole volume fields may be expressed as derivatives of dipole volume fields. The formulas can be derived directly from Eq. 28. For the examples demonstrated here, we assumed a spherical volume conductor; in the case of a realistic head model, derivatives are taken of fields known only numerically. This must be checked for numerical validity when going to increasingly higher order.

We formulated the theory to compute magnetic fields induced by current multipoles. Application to electric potentials, however, is straightforward; one merely has to replace the magnetic dipole fields in Eq. 19 and the magnetic monopole field in Eq. 28 with the respective electric potentials. The contributions from the primary parts in Eq. 28 can be set to zero. Analytical solutions for monopole and dipole potentials can be found in de Munck (1988).

Considering applications of the formulas derived here, a description in terms of current multipoles appears adequate if the brain activity under study is restricted to very few focal regions, as is the case, for example, for the primary somatosensory evoked brain response. In that case the number of degrees of freedom is reasonably small (there is only one expansion point for all multipoles per focal activity), and the series can be expected to converge fast. Using a neurophysiologically plausible model, the measured values of multipole coefficients can finally be used to describe geometric source parameters such as lateral spatial extent; this will be of interest for the comparison of animal results on cerebral neuroplasticity in the somatotopic body representation in the postcentral gyrus (Merzenich et al., 1983; Kaas, 1991) with noninvasive neuroplasticity studies in humans (Mogilner et al., 1993).

APPENDIX A

Here we show that for

$$\hat{f}^{R,I}_{nm}(\nabla) = \frac{(2-\delta_{m0})\cdot(n-m)!}{(n+m)!(2n-1)!!}f^{R,I}_{nm}(\nabla)$$

with $f^R_{nm}(\vec{r}) + if^I_{nm}(\vec{r})$ being the Cartesian form of the polynomial $r^nY^m_n$, the orthonormality relation in Eq. 8 is fulfilled.

The polynomials are given by ($z = r\cos\Theta$),

$$\begin{aligned} r^nY^m_n &= (x+iy)^m r^{n-m}\frac{1}{2^n n!}\left(\frac{\partial}{\partial\cos\Theta}\right)^{n+m}(\cos^2\Theta - 1)^n \\ &= (x+iy)^m d_{nm}(r,z) \end{aligned} \quad \text{(A1)}$$

where d_{nm} is a polynomial in r^2 and z. To get the operators we substitute the Cartesian coordinates by the respective derivatives, and because $r^2 \to \Delta$ and $\Delta r^nY^m_n = 0$, it is sufficient to keep only the highest exponent of z in d.

Now first of all we have $\hat{f}^{R,I}_{nm}f^{R,I}_{\tilde{n}\tilde{m}}|_{x=y=z=0}$ for $n \neq \tilde{n}$, because the order of derivatives in $\hat{f}$ does not match the order of the polynomials in f.

Second, $\hat{f}^{R,I}_{nm}f^{R,I}_{n\tilde{m}}|_{x=y=z=0}$ for $m < \tilde{m}$, because the order of derivatives with respect to z is higher than the highest order of z in f. For $m > \tilde{m}$, there can still be terms left that are of the form $(\partial/\partial z)^k(\partial/\partial x \pm i\partial/\partial y)^{m-\tilde{m}})r^{m-\tilde{m}+k}$, where $m - \tilde{m} + k$ can only be even. Observing that $(\partial/\partial x \pm i\partial/\partial y)r^{2l} = 2lr^{2l-2}(x \pm iy)$ and that $(\partial/\partial x \pm i\partial/\partial y)(x \pm iy) = 0$, the operator can only increase the power of $x \pm iy$ and the term must vanish at the origin.

Third, note that $\hat{f}^R_{nm}f^I_{nm}|_{x=y=z=0} = \hat{f}^I_{nm}f^R_{nm}|_{x=y=z=0} = 0$, because if the derivatives with respect to x are odd, f will contain only even exponents of x and vice versa.

Finally, we compute the normalization N for the real parts (the imaginary parts are identical). For the relevant operator $\hat{f}^R_{nm} = (1/2N)((\partial/\partial x + i\partial/\partial y)^m + (\partial/\partial x - i\partial/\partial y)^m)(\partial/\partial z)^{n-m}$, we have

$$\begin{aligned} \hat{f}^R_{nm}f^R_{nm} &= \frac{1}{2N}\left(\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^m + \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^m\right) \\ &\times \frac{1}{2}((x+iy)^m + (x-iy)^m) \\ &\times \left(\frac{\partial}{\partial z}\right)^{n-m} r^{n-m}\frac{1}{2^n n!}\left(\frac{\partial}{\partial\cos\Theta}\right)^{n+m} \end{aligned}$$

$$(\cos^2\Theta - 1)^n = \frac{1}{N}\frac{m!2^m}{2-\delta_{m0}}\left(\frac{\partial}{\partial z}\right)^{n-m} z^{n-m}\frac{1}{2^n n!}\left(\frac{\partial}{\partial\cos\Theta}\right)^{n+m}$$

$$\begin{aligned} &(\cos^2\Theta - 1)^n|_{\cos\Theta=1} \\ &= \frac{1}{N}\frac{m!2^m}{2-\delta_{m0}}(n-m)!\frac{1}{2^n n!}(n+m)!\binom{n}{m}2^{n-m} \\ &= \frac{1}{N}\frac{(n+m)!}{2-\delta_{m0}} \stackrel{!}{=} 1 \end{aligned} \quad \text{(A2)}$$

which is the second formulation of the operators (Eq. 16). The first formulation (Eq. 15) can be computed by evaluating only the highest exponent of z in $d(r, z)$ and comparing the prefactors.

APPENDIX B

To compute $\vec{B}_{mon}$ for the spherical volume conductor, we place a radial dipole on the z axis at point z_0. Because the total field vanishes, the volume contribution is equal to the negative primary contribution,

$$\vec{B}_{vol}(\vec{r}, z_0\hat{e}_z) = -\frac{\mu_0}{4\pi}\frac{\vec{J}\times(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} \quad \text{(B1)}$$

with $\vec{J} = J\hat{e}_z$ and $\vec{r}_0 = z_0\hat{e}_z$. Because the dipole points in the z direction, $\vec{B}_{mon}$ is found by integrating with respect to z_0. After setting $\mu_0 J/4\pi$ to 1 for proper normalization, as defined in Eq. 24, we find

$$\vec{B}_{mon}(\vec{r}, z_0\hat{e}_z) = \begin{pmatrix}-y\\x\\0\end{pmatrix}\frac{1}{x^2+y^2}\left(\frac{z-z_0}{(x^2+y^2+(z-z_0)^2)^{1/2}} - \frac{z}{r}\right) \quad \text{(B2)}$$

Here we have added an integration constant so that $\vec{B}_{mon}$ is zero if the monopole is located in the center of the sphere. Formally it only has to be radially symmetrical, and one could also add a radially symmetrical field that does not depend on z_0. This field, however, would vanish upon differentiation in any case. Our choice corresponds to taking a reference point located in the center of the sphere. Because the reference point will be unchanged, monopole fields for arbitrary position of the monopole can then be safely found by rotation, using

$$\vec{B}_{mon}(\vec{r}, \vec{r}_0) = U^{-1}\vec{B}_{mon}(U\vec{r}, U\vec{r}_0) \quad \text{(B3)}$$

for an arbitrary rotation matrix U. We choose U to be

$$U = \begin{pmatrix} z_0x_0/r_0(x_0^2+y_0^2)^{1/2} & z_0y_0/r_0(x_0^2+y_0^2)^{1/2} & -(x_0^2+y_0^2)^{1/2}/r_0 \\ -y_0/(x_0^2+y_0^2)^{1/2} & x_0/(x_0^2+y_0^2)^{1/2} & 0 \\ x_0/r_0 & y_0/r_0 & z_0/r_0 \end{pmatrix} \quad \text{(B4)}$$

which has the property $U\vec{r}_0 = (0,0,r_0)^T$. Note that the choice of U is not unique because rotations around the axis monopole to the center of the sphere do not affect $\vec{B}_{mon}$.

Explicit calculation of Eq. B3 results in the final form of the monopole field,

$$\vec{B}_{\text{mon}}(\vec{r}, \vec{r}_0) = \frac{\vec{r}_0 \times \vec{r}}{r^2 r_0^2 - (\vec{r} \cdot \vec{r}_0)^2} \left(\frac{\vec{r} \cdot \vec{r}_0 - r_0^2}{|\vec{r} - \vec{r}_0|} - \frac{\vec{r} \cdot \vec{r}_0}{r} \right) \quad \text{(B5)}$$

We thank W. Haberkorn for thorough reading of the manuscript and many helpful discussions.

Supported by Deutsche Forschungsgemeinschaft Ma 1782/1-1.

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